Relativistic kinetic equation for Compton scattering of polarized radiation in strong magnetic field

Alexander A. Mushtukov

Dmitrij I. Nagirner, Juri Poutanen

6 June 2012

UNIVERSITY of OULU
OULUN YLIOPISTO

Reasons and difficulties

Observations of SGRs and AGNs showed that this objects can be associate with a strong magnetic field. The field can be stronger than Schwinger critical value:

$$B_{\mathrm{cr}} = m_{\mathrm{e}}^2 c^3 / e\hbar = 4.412 \times 10^{13}\ \mathrm{G}$$

Elementary processes can have another behavior in comparison with the case when B-field is weak or absent. Even particles should be described in the another way:

- Electrons occupy Landau levels
- Sometimes radiation in the field can not be described in terms of two polarization modes
- B-field should be taken into account in the calculations of index of reflection in sufficiently strong field $B > 10^{16}\ \mathrm{G}$

Our goals and justification of the method

In this work we focus only on the Compton scattering.

For taking into account multiple scatterings we should use Monte Carlo simulations or use kinetic equations.

We need in kinetic equation which take into account:

- Polarization of radiation in terms of coherency matrix or Stokes parameters
- Induced scattering

There is no recipes for this case **but one can use first principles!**

Particle and field description

Magnetic field: $\vec{B} = B(0,0,1)$ $\quad B > 0$

$$B_{\rm cr} = 4.412 \times 10^{13}\ \mathrm{G} \qquad b = B/B_{\rm cr}$$

Photons: $\underline{k} = \{k, \boldsymbol{k}\},\ k = |\boldsymbol{k}|$

$$\underline{A}_s(\underline{r}) = \underline{e}_s e^{-i\underline{k}\cdot\underline{r}}, \quad \underline{e}_s = \{0, \boldsymbol{e}_s\}, \quad s = 1, 2$$

$$\boldsymbol{e}_1 = (\sin\varphi, -\cos\varphi, 0), \quad \boldsymbol{e}_2 = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

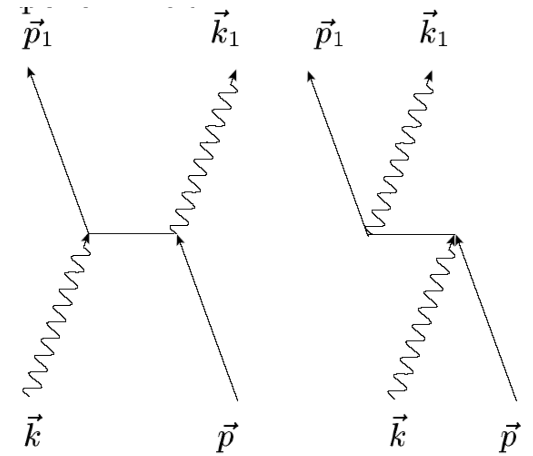
We describe photons in the same manner as in the case when the B-field is absent. This restricts application of the develop formalism to $B \lesssim 10^{16}$ G

The electron states are describes by the wave-functior $\Psi_{n\sigma}(\underline{r}, Y, Z)$
Dimensionless energy of an electron in this case:

$$R_n(Z) = \sqrt{1 + Z^2 + 2bn}.$$

S-matrix and description of a single interaction

There are only 3 conservation laws in a strong B-field:



$$R_{\rm i} + k_{\rm i} = R_{\rm f} + k_{\rm f}$$

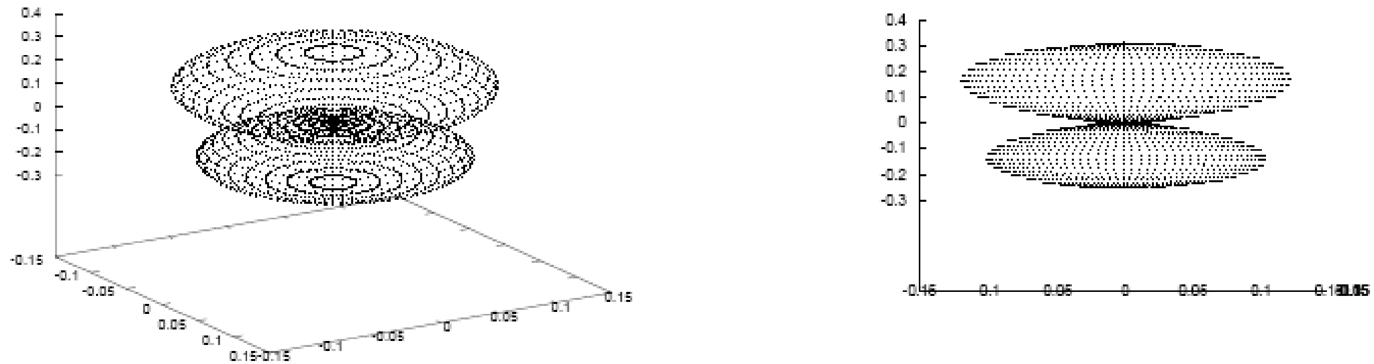
$$Z_{\rm i} + k_{\rm i} \cos\theta_{\rm i} = Z_{\rm f} + k_{\rm f} \cos\theta_{\rm f}$$

$$Y_{\rm i} + k_{\rm i} \sin\theta_{\rm i} \sin\varphi_{\rm i} = Y_{\rm f} + k_{\rm f} \sin\theta_{\rm f} \sin\varphi_{\rm f}$$

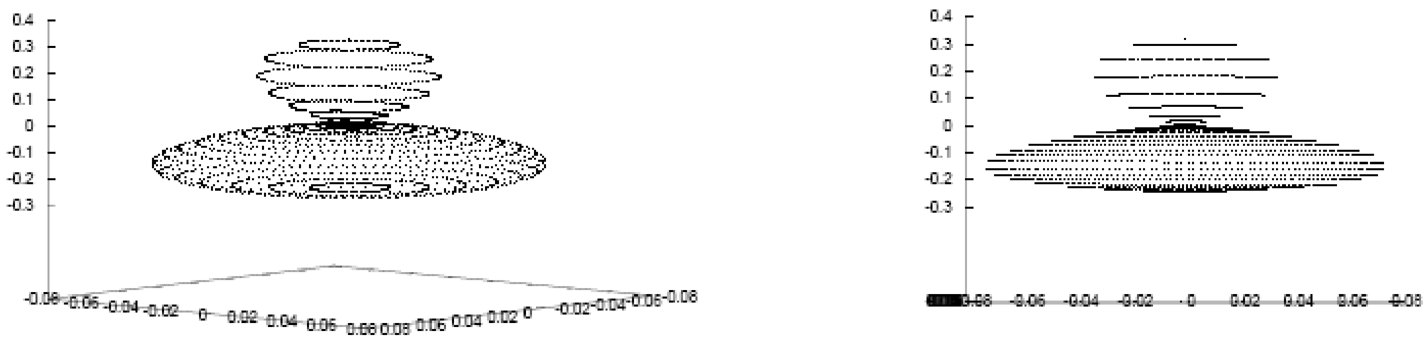
$$S_{\rm f\,i} = -4\pi i\alpha \int {\rm d}^4 r_1 {\rm d}^4 r_2 \overline{\Psi}_{\rm f}(\underline{r}_2) \left\{ \left[\underline{\gamma} \underline{A}_{\rm f}^{\dagger}(\underline{r}_2) \right] G(\underline{r}_2, \underline{r}_1) \left[\underline{\gamma} \underline{A}_{\rm i}(\underline{r}_1) \right] \right.$$
$$\left. + \left[\underline{\gamma} \underline{A}_{\rm i}(\underline{r}_2) \right] G(\underline{r}_2, \underline{r}_1) \left[\underline{\gamma} \underline{A}_{\rm f}^{\dagger}(\underline{r}_1) \right] \right\} \Psi_{\rm i}(\underline{r}_1)$$

From the elements of S-matrix one can find a cross section of the process.

Some numerical results

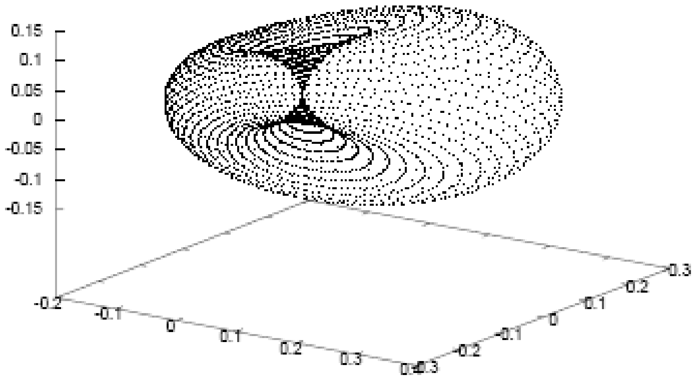


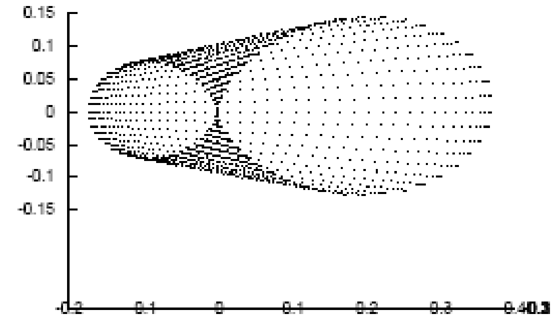
: $k_i = \mathbf{0.04},\ \theta_i = \mathbf{0}\ , n_i = n_f = 1,\ Z_i = 0.01,\ l_i = lf = 2,\ \sigma_i = \sigma_f = 1,\ B = 1.412 * 10^{12}$Гс



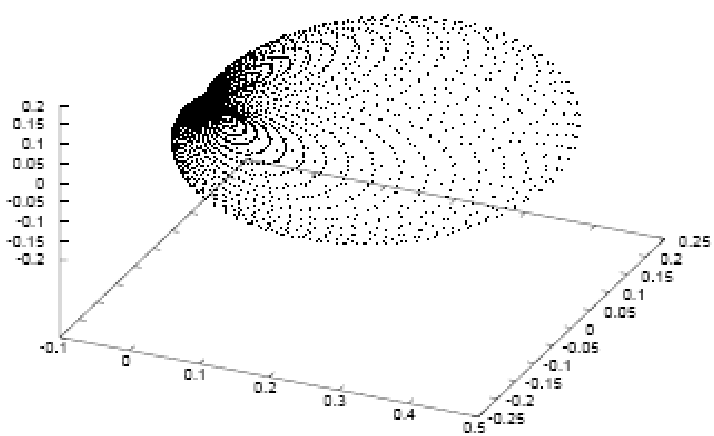
$k_i = \mathbf{1.0},\ \theta_i = \mathbf{0}\ , n_i = n_f = 1,\ Z_i = 0.01,\ l_i = lf = 2,\ \sigma_i = \sigma_f = 1,\ B = 1.412 * 10^{12}$Гс

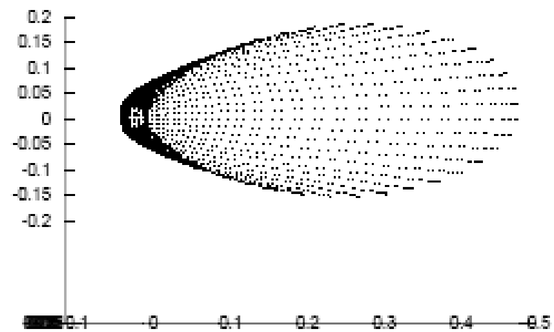
Some numerical results





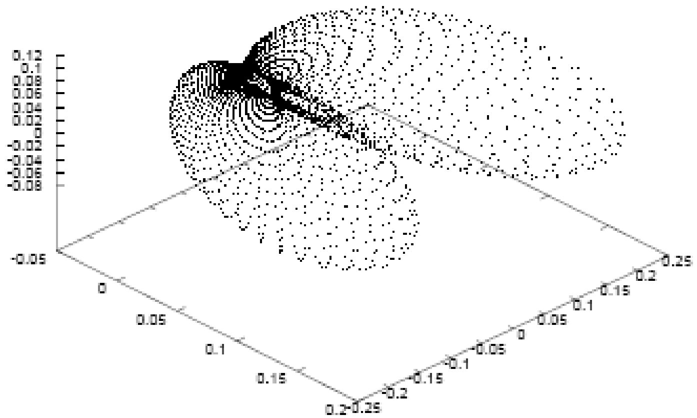
: $k_i = \mathbf{0.07},\ \theta_i = \pi/2\ ,n_i = n_f = 1,\ Z_i = 0.01,\ l_i = lf = 2,\ \sigma_i = \sigma_f = 1,\ B = 1.412 * 10^{12}$Гс

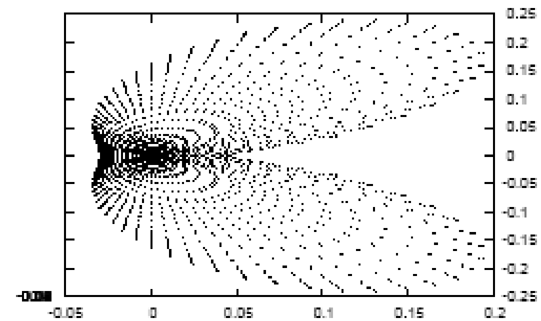


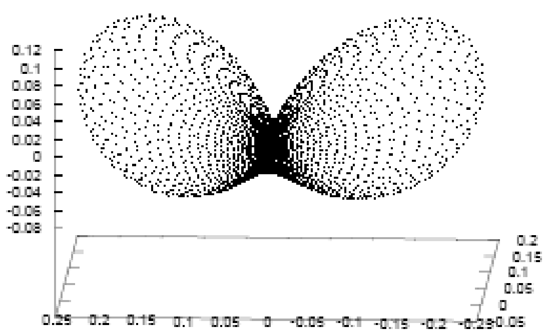


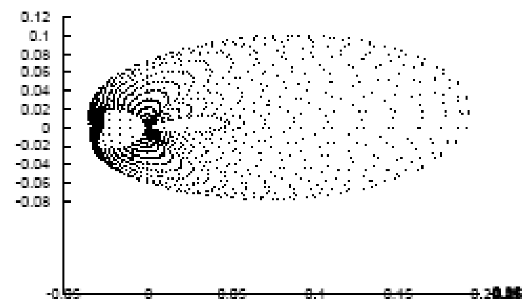
$k_i = \mathbf{0.14},\ \theta_i = \pi/2\ ,n_i = n_f = 1,\ Z_i = 0.01,\ l_i = lf = 2,\ \sigma_i = \sigma_f = 1,\ B = 1.412 * 10^{12}$Гс

Some numerical results

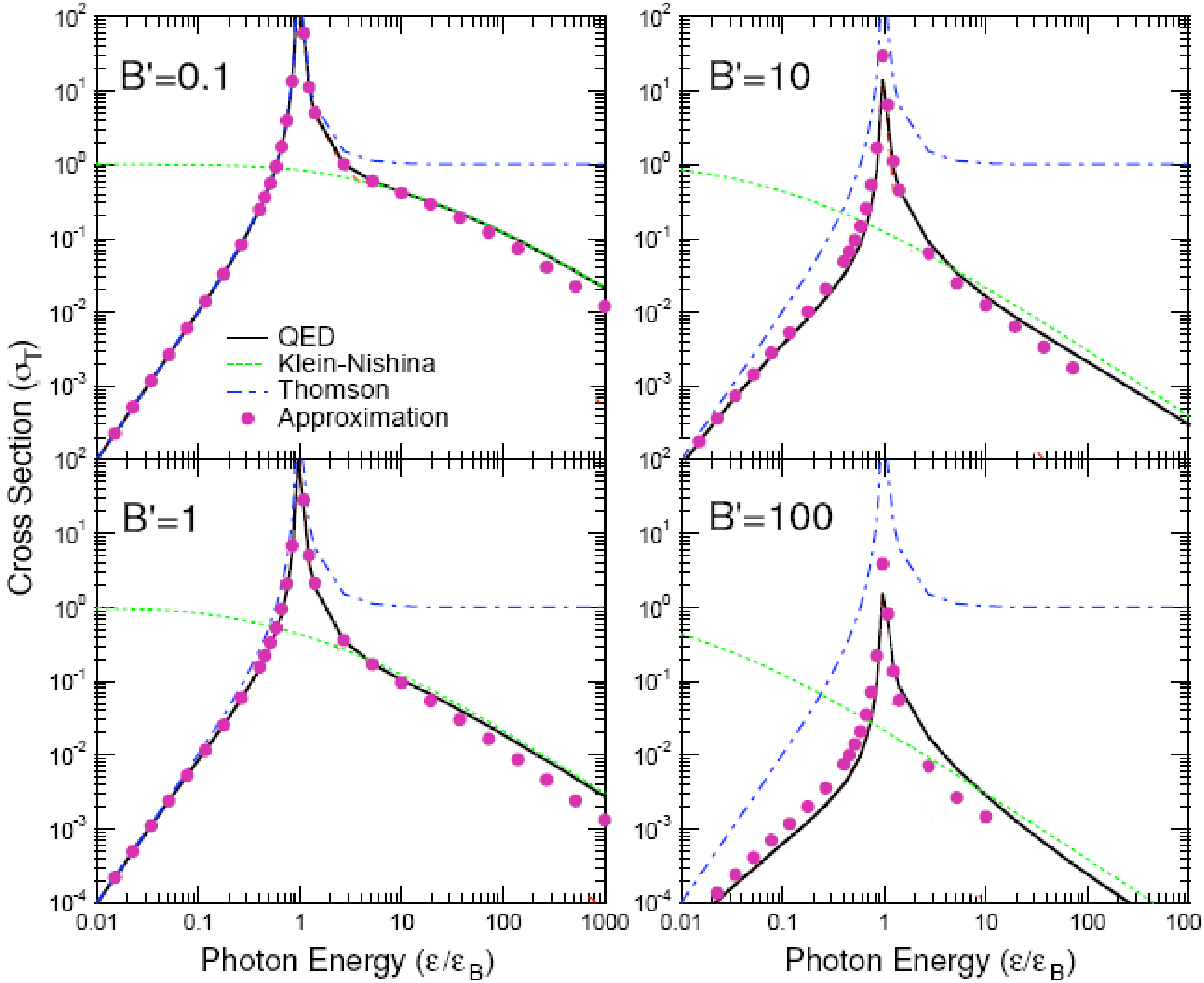








The cross section for CS in magnetic field



Approximations in kinetic theory

- The typical time scales on which the distribution functions changes is mush lager than the typical time scales between the interactions

- The plasma is sufficiently rarefied

- The typical time scales of a single interaction is mush smaller than the typical time scale between the interaction.

- “Molecular chaos” and exchange effects

$$i\hbar \frac{\partial \rho(t)}{\partial t} = H(t)\rho(t) - \rho(t)H(t)$$

Deduction of kinetic equation

$$\rho^{s'_1..s'_N,\sigma'_1...\sigma'_{N_+},n'_1...n'_{N_+}}_{s_1...s_N,\sigma_1...\sigma_{N_+},n_1...n_{N_+}}\left(\begin{matrix}\vec{k}'_1...\vec{k}'_N, Y'_1,...,Y'_{N_+}, Z'_1,...,Z'_{N_+}\\ \vec{k}_1...\vec{k}_N, Y_1,...,Y_{N_+}, Z_1,...,Z_{N_+}\end{matrix}\left|\frac{T_0}{2}\right.\right)=$$

$$=\rho^{s'_1..s'_N,\sigma'_1...\sigma'_{N_+},n'_1...n'_{N_+}}_{s_1..s_N,\sigma_1...\sigma_{N_+},n_1...n_{N_+}}\left(\begin{matrix}\vec{k}'_1...\vec{k}'_N, Y'_1,...,Y'_{N_+}, Z'_1,...,Z'_{N_+}\\ \vec{k}_1...\vec{k}_N, Y_1,...,Y_{N_+}, Z_1,...,Z_{N_+}\end{matrix}\left|-\frac{T_0}{2}\right.\right)+i\frac{e^2}{\hbar c}\frac{mc}{2\pi}\int\frac{dY\,dZ}{R}\frac{dY'dZ'}{R'}\frac{d^3k}{k}\frac{d^3k'}{k'}\delta(*)\times$$

$$\times\, b^{\dagger}_{n'\sigma'}(Y',Z')b_{n\sigma}(Y,Z)\bar{a}_{s'}(\vec{k}')a_s(\vec{k})M^{n'\sigma's'}_{n\sigma s}\left(\begin{matrix}Y',Z',\vec{k}'\\ Y,Z,\vec{k}\end{matrix}\right)\times$$

$$\times\sum_{i=1}^{N}\sum_{i_+=1}^{N_+}\Bigg[\delta^{s'}_{s'_i}\delta(\vec{k}'-\vec{k}'_i)\delta^{n'}_{n'_{i_+}}\delta^{\sigma'}_{\sigma'_{i_+}}\delta(Y'-Y'_{i_+})\delta(Z'-Z'_{i_+})\times$$

$$\times\,\rho^{s'_1...s...s'_N,\sigma'_1...\sigma...\sigma'_{N_+},n'_1...n...n'_{N_+}}_{s_1...s_i...s_N,\sigma_1...\sigma_{i_+}...\sigma_{N_+},n_1...n_{i_+}...n_{N_+}}\left(\begin{matrix}\vec{k}'_1...\vec{k}...\vec{k}'_N, Y'_1,...Y...,Y'_{N_+}, Z'_1,...Z...,Z'_{N_+}\\ \vec{k}_1...\vec{k}_i...\vec{k}_N, Y_1,...Y_{i_+}...,Y_{N_+}, Z_1,...Z_{i_+}...,Z_{N_+}\end{matrix}\left|-\frac{T_0}{2}\right.\right)-$$

$$-\,\delta^{s}_{s_i}\delta(\vec{k}-\vec{k}_i)\delta^{n}_{n_{i_+}}\delta^{\sigma}_{\sigma_{i_+}}\delta(Y-Y_{i_+})\delta(Z-Z_{i_+})\times$$

$$\times\,\rho^{s'_1...s'_i...s'_N,\sigma'_1...\sigma'_{i_+}...\sigma'_{N_+},n'_1...n'_{i_+}...n'_{N_+}}_{s_1...s'...s_N,\sigma_1...\sigma'...\sigma_{N_+},n_1...n'...n_{N_+}}\left(\begin{matrix}\vec{k}'_1...\vec{k}'_i...\vec{k}'_N, Y'_1,...Y'_{i_+}...,Y'_{N_+}, Z'_1,...Z'_{i_+}...,Z'_{N_+}\\ \vec{k}_1...\vec{k}'...\vec{k}_N, Y_1,...Y'...,Y_{N_+}, Z_1,...Z'...,Z_{N_+}\end{matrix}\left|-\frac{T_0}{2}\right.\right)\Bigg].$$

Kinetic equation for photons (general form)

$$\underline{k_1}\underline{\nabla}\rho^{s_1'}_{s_1}(\vec k_1,\vec r_1,t) = i r_0 \frac{m^2c^2}{(2\pi\hbar)^2}\sum_n \int \frac{\mathrm{d}Z}{R}\rho^{\sigma}_{\sigma' n}(Z)\left[\delta^{s'}_{s_1'}\rho^{s}_{s_1}(\vec k_1) M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec k_1\\ nZ & \vec k_1\end{array}\right) - \delta^{s}_{s_1}\rho^{s_1'}_{s'}(\vec k_1) M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec k_1\\ nZ & \vec k\end{array}\right)\right]+$$

$$+ r_0^2 \frac{m^4c^4}{(2\pi\hbar)^3}\sum_{n,n'}\int \frac{\mathrm{d}Z}{R}\frac{\mathrm{d}Z'}{R'}\frac{\mathrm{d}^3k}{k}\delta(k-k_1)\delta(k\cos\theta - k_1\cos\theta_1)\rho^{\sigma}_{\sigma' n}(Z)\rho^{\sigma''}_{\sigma''' n'}(Z')\times$$

$$\times\left[2M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec k_1\\ nZ & \vec k\end{array}\right) M^{\sigma''' s'''}_{\sigma'' s_1}\left(\begin{array}{c|c} n'Z' & \vec k\\ n'Z' & \vec k_1\end{array}\right) \rho^{s}_{s'''}(\vec k) - M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec k_1\\ nZ & \vec k\end{array}\right) M^{\sigma''' s}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec k\\ n'Z' & \vec k_1\end{array}\right)\rho^{s''}_{s_1}(\vec k_1) -\right.$$

$$\left. -M^{\sigma''' s'''}_{\sigma'' s'}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ n'Z' & \vec k\end{array}\right) M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec k\\ nZ & \vec k_1\end{array}\right)\rho^{s_1'}_{s'''}\right]+$$

$$+ r_0^2 \frac{m^4c^4}{(2\pi\hbar)^3}\sum_{n,n'}\int\frac{\mathrm{d}Z}{R}\frac{\mathrm{d}Z'}{R'}\frac{\mathrm{d}^3k}{k}\delta(R+k-R'-k_1)\delta(Z+k\cos\theta - Z' - k_1\cos\theta_1)\times$$

$$\times\left\{\left[\delta^{\sigma'''}_{\sigma}\rho^{\sigma''}_{\sigma' n}(Z) - \delta^{\sigma''}_{\sigma'}\rho^{\sigma'''}_{\sigma n'}(Z')\right]\times\right.$$

$$\times\left[M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec k\\ n'Z' & \vec k_1\end{array}\right) M^{\sigma''' s'''}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ nZ & \vec k\end{array}\right)\rho^{s_1'}_{s'''}(\vec k_1) + M^{\sigma s_1'}_{\sigma' s''}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ nZ & \vec k\end{array}\right) M^{\sigma'' s'}_{\sigma''' s'''}\left(\begin{array}{c|c} nZ & \vec k\\ n'Z' & \vec k_1\end{array}\right)\rho^{s'''}_{s_1}(\vec k_1)\right]\rho^{s''}_{s'}(\vec k)+$$

$$+2M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec k\\ n'Z' & \vec k_1\end{array}\right) M^{\sigma''' s_1'}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ nZ & \vec k\end{array}\right)\rho^{s''}_{s'}(\vec k)\rho^{\sigma''}_{\sigma' n}(Z)\left[\delta^{\sigma'''}_{\sigma} - \rho^{\sigma}_{\sigma''' n'}(Z')\right] -$$

$$-\rho^{\sigma}_{\sigma''' n'}(Z')\left[\delta^{\sigma''}_{\sigma'} - \rho^{\sigma''}_{\sigma' n}(Z)\right]\times$$

$$\left.\times\left[M^{\sigma' s''}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec k\\ n'Z' & \vec k_1\end{array}\right) M^{\sigma''' s'''}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ nZ & \vec k\end{array}\right)\rho^{s_1'}_{s'''}(\vec k_1) + M^{\sigma''' s_1'}_{\sigma' s}\left(\begin{array}{c|c} n'Z' & \vec k_1\\ nZ & \vec k\end{array}\right) M^{\sigma'' s}_{\sigma s''}\left(\begin{array}{c|c} nZ & \vec k\\ n'Z' & \vec k_1\end{array}\right)\rho^{s''}_{s_1}(\vec k_1)\right]\right\}.$$

Kinetic equation for photons (general form)

$$\underline{k}_1\underline{\nabla}\rho^{s_1'}_{s_1}(\vec{k}_1,\vec{r}_1,t) = ir_0\frac{m^2c^2}{(2\pi\hbar)^2}\sum_n\int\frac{\mathrm{d}Z}{R}\rho^{\sigma}_{\sigma' n}(Z)\left[\delta^{s'}_{s_1'}\rho^{s}_{s_1}(\vec{k}_1)M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec{k}_1 \\ nZ & \vec{k}_1 \end{array}\right) - \delta^{s}_{s_1}\rho^{s_1'}_{s'}(\vec{k}_1)M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)\right] +$$

$$+ r_0^2\frac{m^4c^4}{(2\pi\hbar)^3}\sum_{n,n'}\int\frac{\mathrm{d}Z}{R}\frac{\mathrm{d}Z'}{R'}\frac{\mathrm{d}^3k}{k}\delta(k-k_1)\delta(k\cos\theta - k_1\cos\theta_1)\rho^{\sigma}_{\sigma' n}(Z)\rho^{\sigma''}_{\sigma''' n'}(Z')\times$$

$$\times\left[2M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)M^{\sigma''' s'''}_{\sigma'' s_1}\left(\begin{array}{c|c} n'Z' & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)\rho^{s}_{s'''}(\vec{k}) - M^{\sigma' s_1'}_{\sigma s}\left(\begin{array}{c|c} nZ & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)M^{\sigma''' s}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)\rho^{s''}_{s_1}(\vec{k}_1) -\right.$$

$$\left.-M^{\sigma''' s'''}_{\sigma'' s'}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ n'Z' & \vec{k} \end{array}\right)M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec{k} \\ nZ & \vec{k}_1 \end{array}\right)\rho^{s_1'}_{s'''}\right] +$$

$$+ r_0^2\frac{m^4c^4}{(2\pi\hbar)^3}\sum_{n,n'}\int\frac{\mathrm{d}Z}{R}\frac{\mathrm{d}Z'}{R'}\frac{\mathrm{d}^3k}{k}\delta(R+k-R'-k_1)\delta(Z+k\cos\theta - Z' - k_1\cos\theta_1)\times$$

$$\times\left\{\left[\delta^{\sigma'''}_{\sigma}\rho^{\sigma''}_{\sigma' n}(Z) - \delta^{\sigma''}_{\sigma'}\rho^{\sigma'''}_{\sigma n'}(Z')\right]\times\right.$$

$$\times\left[M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)M^{\sigma''' s'''}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)\rho^{s_1'}_{s'''}(\vec{k}_1) + M^{\sigma s_1'}_{\sigma' s''}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)M^{\sigma'' s'}_{\sigma''' s'''}\left(\begin{array}{c|c} nZ & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)\rho^{s'''}_{s_1}(\vec{k}_1)\right]\rho^{s''}_{s'}(\vec{k}) +$$

$$+ 2M^{\sigma' s'}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)M^{\sigma''' s_1'}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)\rho^{s''}_{s'}(\vec{k})\rho^{\sigma''}_{\sigma' n}(Z)\left[\delta^{\sigma'''}_{\sigma} - \rho^{\sigma}_{\sigma''' n'}(Z')\right] -$$

$$-\rho^{\sigma}_{\sigma''' n'}(Z')\left[\delta^{\sigma''}_{\sigma'} - \rho^{\sigma''}_{\sigma' n}(Z)\right]\times$$

$$\left.\times\left[M^{\sigma' s''}_{\sigma s_1}\left(\begin{array}{c|c} nZ & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)M^{\sigma''' s'''}_{\sigma'' s''}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)\rho^{s_1'}_{s'''}(\vec{k}_1) + M^{\sigma''' s_1'}_{\sigma' s}\left(\begin{array}{c|c} n'Z' & \vec{k}_1 \\ nZ & \vec{k} \end{array}\right)M^{\sigma'' s}_{\sigma s''}\left(\begin{array}{c|c} nZ & \vec{k} \\ n'Z' & \vec{k}_1 \end{array}\right)\rho^{s''}_{s_1}(\vec{k}_1)\right]\right\}.$$

Kinetic equation for photons (general form)

The first term describes the redistribution of photons with only changes in polarization and can not be expressed through the cross sections of the process.

The second term describes the redistribution of photons with only changes in polarization but it can be expressed through the cross sections. This term is significant in the case of sufficiently dense electron gas.

The third term describes the general redistribution of photons over energy, direction of motion and polarization.

Kinetic equation for photons (in a special case)

In a case of **non-polarized rarefied electron gas** and **two polarization modes description** the equation have the following form:

$$\underline{k}_1 \underline{\nabla} \rho_{s_1}(\boldsymbol{k}_1, \boldsymbol{r}_1, t) =$$

$$= \frac{\alpha^2}{(2\pi)^3} \sum_{n,n'} \int \frac{\mathrm{d}Y \mathrm{d}Z}{R} \frac{\mathrm{d}Y' \mathrm{d}Z'}{R'} \frac{\mathrm{d}\boldsymbol{k}}{k} \delta(R + k - R' - k_1) \delta(Z + k\cos\theta - Z' - k_1 \cos\theta_1)$$

$$\times T^{s_1 s}_{s \; s_1} \left\{ f_n(Z) \rho_s(\boldsymbol{k}) \left[1 + \rho_{s_1}(\boldsymbol{k}_1)\right] - f_{n'}(Z') \rho_{s_1}(\boldsymbol{k}_1) \left[1 + \rho_s(\boldsymbol{k})\right] \right\}.$$

where: $$T^{ik}_{jm} \equiv M^{\sigma i}_{\sigma' j} \left(\begin{array}{c|c} n'Y'Z' & \boldsymbol{k}_1 \\ nYZ & \boldsymbol{k} \end{array} \right) M^{\sigma' k}_{\sigma m} \left(\begin{array}{c|c} nYZ & \boldsymbol{k} \\ n'Y'Z' & \boldsymbol{k}_1 \end{array} \right)$$

This form of the equation is obvious and can be written immediately using physical arguments.

Results and conclusions

Kinetic equation for Compton scattering was derived. **Polarization of photons** and **spin states of electrons**, the **induced scattering** and **Pauli exclusion principle** were taken into account.
The equation describe the interaction of radiation and electrons in strong B-field up to about 10^{16} G.
There is no low limit on the field strength.

•The equation has a special structure of the *rhs* and this structure can be found only in the case when one derive the equation from the first principles.
•From the structure of equation one can conclude that the effects of rotation of polarization plane can exist in a special regions in the atmospheres of neutron stars.
•The effects can play a role in a mixing of O- and X-modes.

Thank you

for your attention!